Parametric E-differentiable multiobjective fractional programming under E-invexity

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Abstract— In this paper, we consider a new class of (not necessarily) differentiable multiobjective fractional programming problems with both inequality and equality constraints in which the functions involved are E-differentiable. The so-called parametric E-Karush-Kuhn-Tucker necessary optimality conditions are established. Further, the parametric sufficient optimality conditions are derived for the considered E-differentiable multiobjective fractional programming problems under E-invexity hypotheses.

Index Terms— E-differentiable parametric multiobjective fractional programming; E-invex function; parametric E-optimality conditions.
AMS Classification: 90C26; 90C29; 90C32; 90C46; 90C47.

1 Introduction

In this paper, we consider the following (not necessarily differentiable) multiobjective fractional programming problem (MFP) with both inequality and equality constraints:

$$\begin{aligned} &\min && \left(\frac{f_1(x)}{q_1(x)},\ldots,\frac{f_p(x)}{q_p(x)}\right) \\ &\text{subjectto} && g_j(x) \leqq 0, \quad j=1,\ldots,m, \\ &&& h_t(x)=0, \quad t=1,\ldots,k, \end{aligned} \qquad \text{(MFP)}$$

where $f_i: R^n \to R$, $q_i: R^n \to R$, $i \in I = \{1,\ldots,p\}$, $g_j: R^n \to R$, $j \in J = \{1,\ldots,m\}$, $h_t: R^n \to R$, $t \in T = \{1,\ldots,k\}$, are (not necessarily) differentiable functions on R^n.

Further, we shall assume that $f_i(x) \geqq 0$, $i \in I$, $q_i(x) > 0$, $i \in I$, for all $x \in R^n$. Let $\Omega := \{x \in R^n : g_j(x) \leqq 0,\ j \in J, h_t(x) = 0, t \in T\}$ be the set of all feasible solutions in (MFP).

In the past few years, multiobjective fractional programming problems have attracted the attention of many researchers due to the fact that in many operations research problems the objective functions are quotients of two functions. Therefore, many authors have introduced various concepts of generalized convexity and have established optimality conditions and duality results for multiobjective fractional programming problems (see, for example, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], and others).

The concept of convexity plays a vital real in proving the fundamental results in optimization theory. Therefore, various classes of nonconvex (not necessarily) differentiable optimization problems have been defined in optimization theory. One of such important generalizations of the convexity notion is the concept of E-convexity introduced by Youness [26]. This kind of generalized convexity is based on the effect of an operator $E: R^n \to R^n$ on the sets and the domain of the definition of functions. Recently, Megahed et al. [27] presented the concept of an E-differentiable convex function which transforms a (not necessarily) differentiable convex function to a differentiable function also based on the effect of an operator $E: R^n \to R^n$. Antczak and Abdulaleem [28] proved the so-called E-optimality conditions and Wolfe E-duality for E-differentiable vector optimization problems with both inequality and equality constraints. Recently, Abdulaleem [1] introduced a new concept of generalized convexity as a generalization of the notion of E-differentiable E-convexity. Namely, he defined the concept of E-differentiable E-invexity in the case of (not necessarily) differentiable vector optimization problems with E-differentiable functions.

In this paper, we consider a new class of (not necessarily) differentiable multiobjective fractional programming problems in which the involved functions are E-differentiable. We prove the

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parametric necessary E-optimality conditions for the considered E-differentiable multiobjective fractional programming problem. Further, we prove sufficient E-optimality conditions for the considered E-differentiable multiobjective fractional programming problem under E-invexity hypotheses.

2 Definitions and preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. The following convention for equalities and inequalities will be used in the paper. For any vectors $x=(x_1,x_2,\dots,x_n)^T$ and $y=(y_1,y_2,\dots,y_n)^T$ in R^n, we define:

- $x=y$ if and only if $x_i=y_i$ for all $i=1,2,\cdots,n$;
- $x>y$ if and only if $x_i>y_i$ for all $i=1,2,\cdots,n$;
- $x\geqq y$ if and only if $x_i\geqq y_i$ for all $i=1,2,\cdots,n$;
- $x\geq y$ if and only if $x_i\geqq y_i$ for all $i=1,2,\cdots,n$ but $x\neq y$.

We now give the definitions of an E-invex set and an E-invex function introduced by Abdulaleem [1].

Definition 1 [1] *Let $E: R^n\to R^n$. A set $M\subseteq R^n$ is said to be an E-invex set if and only if there exists a vector-valued function $\eta: M\times M\to R^n$ such that the relation*

$$E(u)+\lambda\eta\big(E(x),E(u)\big)\in M$$

holds for all $x,u\in M$ and any $\lambda\in[0,1]$.

Definition 2 [27] *Let $E: R^n\to R^n$ and $f: R^n\to R$ be a (not necessarily) differentiable function at a given point $u\in R^n$. It is said that f is an E-differentiable function at u if and only if $f\circ E$ is a differentiable function at u (in the usual sense), that is,*

$$(f\circ E)(x)=(f\circ E)(u)+\nabla(f\circ E)(u)(x-u)+\theta(u,x-u)\|x-u\|, \quad (1)$$

where $\theta(u,x-u)\to 0$ as $x\to u$.

Definition 3 [1] *Let $E: R^n\to R^n$, $M\subseteq R^n$ be a nonempty open E-invex set with respect to the vector-valued function $\eta: R^n\times R^n\to R^n$ and $f: R^n\to R^k$ be an E-differentiable function on M. It is said that f is a vector-valued (strictly) E-invex function with respect to η at u on M if, for all $x\in M$, $i=1,\dots,k$,*

$$f_i\big(E(x)\big)-f_i\big(E(u)\big)\geqq \nabla(f_i\circ E)(u)\eta(E(x),E(u))\ (>). \quad (2)$$

If inequalities (2) hold for any $u\in M$, then f is (strictly) E-invex with respect to η on M.

Remark 4 From Definition 3, there are the following special cases:
1. If f is a differentiable function and $E(x)\equiv x$ (E is an identity map), then the definition of an E-invex function reduces to the definition of an invex function introduced by Hanson [29].
2. If $\eta: R^n\times R^n\to R^n$ is defined by $\eta(x,u)=x-u$, then we obtain the definition of an E-differentiable E-convex vector-valued function introduced by Megahed et al. [27].
3. If f is differentiable, $E(x)=x$ and $\eta(x,u)=x-u$, then the definition of an E-invex function reduces to the definition of a differentiable convex vector-valued function.
4. If f is differentiable and $\eta(x,u)=x-u$, then we obtain the definition of a differentiable E-convex function introduced by Youness [26].

3 E-differentiable multiobjective fractional programming

Let $E: R^n\to R^n$ be a given one-to-one and onto operator. Now, for the considered E-differentiable multiobjective fractional programming problem (MFP), we define its associated differentiable multiobjective fractional programming problem (MFP $_E$) as follows:

$$\begin{aligned} \min\quad & \left(\frac{f_1\big(E(x)\big)}{q_1\big(E(x)\big)},\dots,\frac{f_p\big(E(x)\big)}{q_p\big(E(x)\big)}\right) \\ \text{subject to}\quad & g_j\big(E(x)\big)\leqq 0,\quad j=1,\dots,m, \\ & h_t\big(E(x)\big)=0,\quad t=1,\dots,k, \end{aligned} \quad (\mathrm{MFP}_E)$$

where the functions $f_i\circ E: R^n\to R$, $q_i\circ E: R^n\to R$, $i=1,2,\dots,p$, $g_j\circ E: R^n\to R$, $j=1,2,\dots,m$, $h_t\circ E: R^n\to R$, $t=1,2,\dots,k$, are differentiable real-valued functions. Let $\Omega_E:=\{x\in R^n: (g_j\circ E)(x)\leqq 0,\ j\in J, (h_t\circ E)(x)=0, t\in T\}$ be the set of all feasible solutions of (MFP $_E$).

Lemma 5 [28] *Let $E: R^n\to R^n$ be a one-to-one and onto operator. Then $E(\Omega_E)=\Omega$.*

Let us denote $\Phi=\big(\Phi_1,\dots,\Phi_p\big): R^n\to R^p$, where $\Phi_i(z)=\frac{f_i(z)}{q_i(z)}$, $i\in I$.

Definition 6 *$\overline{x} \in \Omega_E$ is said to be a weak Pareto solution (a weakly efficient solution) of (MFP $_E$) if and only if there is no another $x \in \Omega_E$ such that*

$$\Phi(E(x)) < \Phi(E(\overline{x})). \tag{3}$$

Definition 7 *$\overline{x} \in \Omega_E$ is said to be a Pareto solution (an efficient solution) of (MFP $_E$) if and only if there is no another $x \in \Omega_E$ such that*

$$\Phi(E(x)) \leq \Phi(E(\overline{x})). \tag{4}$$

Definition 8 *$E(\overline{x}) \in \Omega$ is said to be a weak E-Pareto solution (a weakly E-efficient solution) of (MFP) if and only if there is no another feasible solution $E(x) \in \Omega$ such that (3) is satisfied.*

Definition 9 *$E(\overline{x}) \in \Omega$ is said to be an E-Pareto solution (an E-efficient solution) of (MFP) if and only if there is no another feasible solution $E(x) \in \Omega$ such that (4) is satisfied.*

Proposition 10 [28] *Let $E: R^n \to R^n$ be a one-to-one and onto operator. $\overline{x} \in \Omega_E$ is said to be a weak Pareto solution (a Pareto solution) of (MFP $_E$) if and only if $E(\overline{x}) \in \Omega$ is said to be a weak E-Pareto solution (an E-Pareto solution) of (MFP).*

Therefore, for the foregoing multiobjective fractional programming problems (MFP) and (MFP $_E$), we define their associated nonfractional parametric vector optimization problem (MP $^{\lambda}_E$) for $\lambda^E = (\lambda_1^E, \ldots, \lambda_p^E) \in R^p$ as follows

$$\min \left(f_1(E(x)) - \lambda_1^E q_1(E(x)), \ldots, f_p(E(x)) - \lambda_p^E q_p(E(x)) \right)$$

$$\text{subject to} \quad g_j(E(x)) \leqq 0, \quad j = 1, \ldots, m, \qquad (\text{MP}_E^{\lambda^E})$$
$$h_t(E(x)) = 0, \quad t = 1, \ldots, k.$$

Note that the set of feasible solutions of the nonfractional parametric vector optimization problem (MP $^{\lambda^E}_E$) is the same as the set of all feasible solutions of (MFP $_E$)

For the auxiliary multiobjective programming problem (MP $^{\lambda^E}_E$) defined above, the following results are true:

Lemma 11 *$\overline{x}$ is a weakly efficient solution (an efficient solution) of the multiobjective fractional programming problem (MFP $_E$) if and only if $\overline{x}$ is a weakly efficient solution (an efficient solution) of the problem (MP $^{\lambda^E}_E$), where $\overline{\lambda}_i^E = \frac{f_i(E(\overline{x}))}{q_i(E(\overline{x}))}$, $i \in I$.*

Lemma 12 *$E(\overline{x})$ is a weakly E-efficient solution (an E-efficient solution) of the considered multiobjective fractional programming problem (MFP) if and only if $\overline{x}$ is a weakly efficient solution (an efficient solution) of the problem (MP $^{\lambda^E}_E$), where $\overline{\lambda}_i^E = \frac{f_i(E(\overline{x}))}{q_i(E(\overline{x}))}$, $i \in I$.*

Theorem 13 *(Parametric necessary optimality conditions for (MFP $_E$)). Let $\overline{x} \in \Omega_E$ be a weak Pareto solution of the multiobjective fractional programming problem (MFP $_E$)) with $\overline{\lambda}^E = \Phi(E(\overline{x}))$. Further, assume that the Guignard constraint qualification* [30] *is satisfied at $\overline{x}$. Then, there exist $\overline{\tau} \in R^p$, $\overline{\zeta} \in R^m$, $\overline{\mu} \in R^k$ such that the following conditions are satisfied:*

$$\sum_{i=1}^p \overline{\tau}_i \nabla \left(f_i(E(\overline{x})) - \overline{\lambda}_i^E q_i(E(\overline{x})) \right) + \sum_{j=1}^m \overline{\zeta}_j \nabla g_j(E(\overline{x})) + \sum_{t=1}^k \overline{\mu}_t \nabla h_t(E(\overline{x})) = 0, \tag{5}$$

$$\overline{\zeta}_j g_j(E(\overline{x})) = 0, \quad j = 1, \ldots, m, \tag{6}$$

$$\overline{\tau}_i \geqq 0, \ i \in I, \ \sum_{i=1}^p \overline{\tau}_i = 1, \ \overline{\zeta}_j \geqq 0, \ j \in J. \tag{7}$$

Proof. Let $\overline{x} \in D_E$ be a weak Pareto solution of the multiobjective fractional programming problem (MFP $_E$). Hence, by Lemma 11, $\overline{x} \in D_E$ is also a weak Pareto solution of its associated nonfractional parametric multiobjective programming problem (MP $^{\lambda^E}_E$). Further, by Theorem 38 [30], there exist Lagrange multipliers $\overline{\tau} \in R^p$, $\overline{\zeta} \in R^m$, $\overline{\mu} \in R^k$ such that the above conditions (5)-(7) are fulfilled. This completes the proof of this theorem. ■

Theorem 14 *(Parametric necessary E-optimality conditions for (MFP)). Let $E(\overline{x})$ be a weak E-Pareto solution of the considered multiobjective fractional programming problem (MFP). Further, assume that the Guignard constraint qualification* [30] *is satisfied at $\overline{x}$. Then, there exist $\overline{\tau} \in R^p$, $\overline{\zeta} \in R^m$, $\overline{\mu} \in R^k$ such that the conditions (5)-(7) are satisfied.*

Theorem 15 *(Sufficient optimality conditions for (MFP $_E$)). Let $\overline{x}$ be a feasible solution of the multiobjective fractional E-programming problem (MFP $_E$), $\overline{\lambda}_i^E = \frac{f_i(E(\overline{x}))}{q_i(E(\overline{x}))}$, $i \in I$, and the parametric necessary optimality conditions (5)-(7) be satisfied at $\overline{x}$*

with the Lagrange multipliers $\overline{\tau} \in R^p$, $\overline{\zeta} \in R^m$ and $\overline{\mu} \in R^k$. Further, assume that the following hypotheses are fulfilled:

a) each function f_i, $i \in I$, is (strictly) E-invex with respect to η at $\overline{x}$ on Ω_E,
b) each function $-q_i$, $i \in I$, is E-invex with respect to η at $\overline{x}$ on Ω_E,
c) each function g_j, $j \in J_E(\overline{x})$, is E-invex with respect to η at $\overline{x}$ on Ω_E,
d) each function h_t, $t \in T_E^+(\overline{x}) := \{t \in T : \overline{\mu} > 0\}$, is E-invex with respect to η at $\overline{x}$ on Ω_E,
e) each function $-h_t$, $t \in T_E^-(\overline{x}) := \{t \in T : \overline{\mu} < 0\}$, is E-invex with respect to η at $\overline{x}$ on Ω_E.

Then $\overline{x}$ is (a weakly efficient solution) an efficient solution of the multiobjective fractional E-programming problem (MFP $_E$) and, at the same time, $E(\overline{x})$ is (a weakly E-efficient solution) an E-efficient solution of the original E-differentiable multiobjective fractional programming problem (MFP).

Proof. Let the necessary optimality conditions (5)-(7) be satisfied at $\overline{x} \in \Omega$ with Lagrange multipliers $\overline{\tau} \in R^p$, $\overline{\zeta} \in R^m$ and $\overline{\mu} \in R^k$. Suppose, contrary to the result, that $\overline{x}$ is not an efficient solution of the problem (MFP $_E$). Hence, by Definition 6, there exists $\hat{x} \in \Omega_E$ such that

$$\frac{f_i(E(\hat{x}))}{q_i(E(\hat{x}))} \leqq \frac{f_i(E(\overline{x}))}{q_i(E(\overline{x}))}, \qquad i \in I,$$

$$\frac{f_{i^\star}(E(\hat{x}))}{q_{i^\star}(E(\hat{x}))} < \frac{f_{i^\star}(E(\overline{x}))}{q_{i^\star}(E(\overline{x}))}$$

forsome $i^\star \in I$. Hence, the inequalities above imply

$$f_i(E(\hat{x})) - \overline{\lambda}_i^E q_i(E(\hat{x})) \leqq f_i(E(\overline{x})) - \overline{\lambda}_i q_i(E(\overline{x})), \quad i \in I, \tag{8}$$

$$f_{i^\star}(E(\hat{x})) - \overline{\lambda}_{i^\star}^E q_{i^\star}(E(\hat{x})) < f_{i^\star}(E(\overline{x})) - \overline{\lambda}_{i^\star} q_{i^\star}(E(\overline{x})) \tag{9}$$

forsome $i^\star \in I$. Adding both sides of the inequalities (8) and (9), we get

$$\sum_{i=1}^{p} \overline{\tau}_i \left[f_i(E(\hat{x})) - \overline{\lambda}_i^E q_i(E(\hat{x}))\right] < \sum_{i=1}^{p} \overline{\tau}_i \left[f_i(E(\overline{x})) - \overline{\lambda}_i^E q_i(E(\overline{x}))\right]. \tag{10}$$

Using the assumptions a)-b), we have, by Definition 3, that the inequalities

$$f_i(E(x)) - f_i(E(\overline{x})) > \nabla(f_i \circ E)(\overline{x})\eta(E(x), E(\overline{x})), \quad i \in I, \tag{11}$$

$$-q_i(E(x)) + q_i(E(\overline{x})) \geqq -\nabla(q_i \circ E)(\overline{x})\eta(E(x), E(\overline{x})), \quad i \in I, \tag{12}$$

$$g_j(E(x)) - g_j(E(\overline{x})) \geqq \nabla(g_j \circ E)(\overline{x})\eta(E(x), E(\overline{x})), \quad j \in J(\overline{x}), \tag{13}$$

$$h_t(E(x)) - h_t(E(\overline{x})) \geqq \nabla(h_t \circ E)(\overline{x})\eta(E(x), E(\overline{x})), t \in T_E^+(\overline{x}), \tag{14}$$

$$-h_t(E(x)) + h_t(E(\overline{x})) \geqq -\nabla(h_t \circ E)(\overline{x})\eta(E(x), E(\overline{x})), t \in T_E^-(\overline{x}) \tag{15}$$

hold for all $x \in \Omega_E$. Therefore, they are also satisfied for $x = \hat{x}$. Thus,

$$f_i(E(\hat{x})) - f_i(E(\overline{x})) > \nabla(f_i \circ E)(\overline{x})\eta(E(\hat{x}), E(\overline{x})), \quad i \in I, \tag{16}$$

$$-q_i(E(\hat{x})) + q_i(E(\overline{x})) \geqq -\nabla(q_i \circ E)(\overline{x})\eta(E(\hat{x}), E(\overline{x})), \quad i \in I, \tag{17}$$

$$g_j(E(\hat{x})) - g_j(E(\overline{x})) \geqq \nabla(g_j \circ E)(\overline{x})\eta(E(\hat{x}), E(\overline{x})), \quad j \in J_E(\overline{x}), \tag{18}$$

$$h_t(E(\hat{x})) - h_t(E(\overline{x})) \geqq \nabla(h_t \circ E)(\overline{x})\eta(E(\hat{x}), E(\overline{x})), \quad t \in T_E^+(\overline{x}), \tag{19}$$

$$-h_t(E(\hat{x})) + h_t(E(\overline{x})) \geqq -\nabla(h_t \circ E)(\overline{x})\eta(E(\hat{x}), E(\overline{x})), \ t \in T_E^-(\overline{x}) \tag{20}$$

We multiple the inequalities above by corresponding Lagrange multipliers and, moreover, we multiply (17) extra by $\overline{\lambda}_i^E = \frac{f_i(E(\overline{x}))}{q_i(E(\overline{x}))} \geqq 0$, $i \in I$. After summing the resulting inequalities and taking into account Lagrange multipliers equal to 0, (16)-(20) yield

$$\sum_{i=1}^{p} \overline{\tau}_i \left[f_i(E(\hat{x})) - \overline{\lambda}_i^E q_i(E(\hat{x}))\right] + \sum_{j=1}^{m} \overline{\zeta}_j g_j(E(\hat{x})) + \sum_{t=1}^{k} \overline{\mu}_t h_t((E(\hat{x})) - \sum_{i=1}^{p} \overline{\tau}_i \left[f_i(E(\overline{x})) - \overline{\lambda}_i^E q_i(E(\overline{x}))\right] + \sum_{j=1}^{m} \overline{\zeta}_j g_j(E(\overline{x})) + \sum_{t=1}^{k} \overline{\mu}_t h_t((E(\overline{x})) \geqq \Big[\sum_{i=1}^{p} \overline{\tau}_i \nabla(f_i \circ E)(\overline{x}) - \overline{\lambda}_i^E \nabla(q_i \circ E)(\overline{x}) + \sum_{j=1}^{m} \overline{\zeta}_j \nabla(g_j \circ E)(\overline{x}) + \sum_{t=1}^{k} \overline{\mu}_t \nabla(h_t \circ E)(\overline{x})\Big]\eta(E(\hat{x}), E(\overline{x})). \tag{21}$$

Hence, by the parametric necessary optimality condition (5), (21) implies

$$\sum_{i=1}^{p} \overline{\tau}_i \left[f_i(E(\hat{x})) - \overline{\lambda}_i^E q_i(E(\hat{x}))\right] +$$

$$\sum_{j=1}^{m} \overline{\zeta}_j g_j\big(E(\hat{x})\big) + \sum_{t=1}^{k} \overline{\mu}_t h_t(\big(E(\hat{x})\big) \geqq \sum_{i=1}^{p} \overline{\tau}_i \left[f_i\big(E(\overline{x})\big) - \overline{\lambda}_i^E q_i\big(E(\overline{x})\big)\right] + \sum_{j=1}^{m} \overline{\zeta}_j g_j\big(E(\overline{x})\big) + \sum_{t=1}^{k} \overline{\mu}_t h_t(\big(E(\overline{x})\big).$$

Using by the parametric necessary optimality condition (6) together with the feasibility of $\hat{x}$ and $\overline{x}$ in the problem (MFP $_E$), we have that the inequality

$$\sum_{i=1}^{p} \overline{\tau}_i \left[f_i\big(E(\hat{x})\big) - \overline{\lambda}_i^E q_i\big(E(\hat{x})\big)\right] \geqq \sum_{i=1}^{p} \overline{\tau}_i \left[f_i\big(E(\overline{x})\big) - \overline{\lambda}_i^E q_i\big(E(\overline{x})\big)\right]$$

holds, contradicting (10). The proof in the case of weakly E-efficiency is similar and, therefore, it has been omitted in the paper. Thus, the proof of this theorem is completed. ■

4 Conclusions

In this paper, the class of E-differentiable multiobjective fractional programming problems has been considered. The parametric E-Karush-Kuhn-Tucker necessary optimality conditions have been established. Further, for the considered E-differentiable vector optimization problems, the parametric sufficient E-optimality conditions have been established under appropriate E-invexity hypotheses. The E-optimality established in the present paper generalize and extend similar results for multiobjective fractional programming problems with E-differentiable functions presented in earlier works.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results under E-invexity hypotheses for other classes of E-differentiable fractional vector optimization problems. We shall investigate these questions in subsequent papers.

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